Physics 513, Quantum Field Theory

Homework 4

Due Tuesday, 30th September 2003

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1. We have defined the *coherent state* by the relation

$$|\{\eta_k\}\rangle \equiv \mathcal{N} \exp\left\{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}\right\}|0\rangle.$$

For my own personal convenience throughout this solution, I will let

$$\mathcal{A} \equiv \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k^\dagger}{\sqrt{2E_k}}.$$

a) Lemma: $\left[a_p, e^{\mathcal{A}}\right] = \frac{\eta_p}{\sqrt{2E_p}}e^{\mathcal{A}}$.

proof: First we note that from simple Taylor expansion (which is justified here),

$$e^{\mathcal{A}} = 1 + \mathcal{A} + \frac{\mathcal{A}^2}{2} + \frac{\mathcal{A}^3}{3!} + \dots$$

Clearly a_p commutes with 1 so we may write,

$$\begin{aligned}\left[a_p, e^{\mathcal{A}}\right] &= [a_p,\mathcal{A}] + \frac{1}{2}[a_p,\mathcal{A}^2] + \frac{1}{3!}[a_p,\mathcal{A}^3] + \dots,\\ &= [a_p,\mathcal{A}] + \frac{1}{2}\left([a_p,\mathcal{A}]\mathcal{A} + \mathcal{A}[a_p,\mathcal{A}]\right) + \frac{1}{3!}\left([a_p,\mathcal{A}]\mathcal{A}^2 + \mathcal{A}[a_p,\mathcal{A}]\mathcal{A} + \mathcal{A}[a_p,\mathcal{A}]\mathcal{A}\right) + \dots,\\ &\overset{*}{=} [a_p,\mathcal{A}]\left(1 + \mathcal{A} + \frac{\mathcal{A}^2}{2} + \frac{\mathcal{A}^3}{3!} + \frac{\mathcal{A}^4}{4!} + \dots\right),\\ &= [a_p,\mathcal{A}]e^{\mathcal{A}}.\end{aligned}$$

Note that the step labelled '*' is unjustified. To allow the use of '*' we must show that $[a_p,\mathcal{A}]$ is an invariant scalar and therefore commutes with all the $\mathcal{A}$'s. This is shown by direct calculation.

$$\begin{aligned}[a_p,\mathcal{A}] &= \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k}{\sqrt{2E_k}}[a_p, a_k^\dagger],\\ &= \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k}{\sqrt{2E_k}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k}),\\ &= \frac{\eta_p}{\sqrt{2E_p}}.\end{aligned}$$

This proves what was required for '*.' $\frac{\eta_p}{\sqrt{2E_p}}$ is clearly a scalar because η and E_p are real numbers only. But by demonstrating the value of $[a_p,\mathcal{A}]$ we can complete the proof of the required lemma. Clearly,

$$\left[a_p, e^{\mathcal{A}}\right] = [a_p,\mathcal{A}]e^{\mathcal{A}} = \frac{\eta_p}{\sqrt{2E_p}}e^{\mathcal{A}}.$$

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It is clear from the definition of the commutator that $a_p e^{\mathcal{A}} = \left[a_p, e^{\mathcal{A}}\right] + e^{\mathcal{A}}a_p$. Therefore it is intuitively obvious, and also proven that

$$\begin{aligned}a_p|\{\eta_k\}\rangle &= \mathcal{N}a_p e^{\mathcal{A}}|0\rangle,\\ &= \mathcal{N}\left(\left[a_p, e^{\mathcal{A}}\right] + e^{\mathcal{A}}a_p\right)|0\rangle,\\ &= \mathcal{N}\frac{\eta_p}{\sqrt{2E_p}}|0\rangle + \mathcal{N}e^{\mathcal{A}}a_p|0\rangle,\\ \therefore a_p|\{\eta_k\}\rangle &= \frac{\eta_p}{\sqrt{2E_p}}a_p|\{\eta_k\}\rangle. \qquad (1.1)\end{aligned}$$

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b) We are to compute the normalization constant $\mathcal{N}$ so that $\langle\{\eta_k\}|\{\eta_k\}\rangle = 1$. I will proceed by direct calculation.

$$\begin{aligned}
1 &= \langle\{\eta_k\}|\{\eta_k\}\rangle,\\
&= \mathcal{N}^*\langle 0|e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k a_k}{\sqrt{2E_k}}}|\{\eta_k\}\rangle,\\
&= \mathcal{N}^*\langle 0|e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k}{\sqrt{2E_k}}}|\{\eta_k\}\rangle
\end{aligned}$$

because we know that $a_k|\{\eta_k\}\rangle = \frac{\eta_k}{\sqrt{2E_k}}|\{\eta_k\}\rangle$. So clearly

$$\begin{aligned}
1 &= |\mathcal{N}|^2 e^{\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}},\\
\therefore \mathcal{N} &= e^{-\frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}}.
\end{aligned}$$

c) We will find the expectation value of the field $\phi(x)$ by direct calculation as before.

$$\begin{aligned}
\overline{\phi(x)} = \langle\{\eta_k\}|\phi(x)|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a_p e^{i\vec{p}\cdot\vec{x}} + a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}\right)|\{\eta_k\}\rangle,\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\underbrace{\langle\{\eta_k\}|a_p e^{i\vec{p}\cdot\vec{x}}|\{\eta_k\}\rangle}_{\text{act with } a_p \text{ to the right}} + \underbrace{\langle\{\eta_k\}|a_p^\dagger e^{-i\vec{p}\cdot\vec{x}}|\{\eta_k\}\rangle}_{\text{act with } a_p^\dagger \text{ to the left}}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(\frac{\eta_p}{\sqrt{2E_p}}e^{i\vec{p}\cdot\vec{x}} + \frac{\eta_p}{\sqrt{2E_p}}e^{-i\vec{p}\cdot\vec{x}}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_p}{E_p}\cos(\vec{p}\cdot\vec{x}).
\end{aligned}$$

d) We will compute the expected particle number directly.

$$\begin{aligned}
\overline{N} = \langle\{\eta_k\}|N|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3p}{(2\pi)^3}a_p^\dagger a_p|\{\eta_k\}\rangle,\\
&= \int \frac{d^3p}{(2\pi)^3}\left(\langle\{\eta_k\}|\underleftarrow{a_p^\dagger}\ \underrightarrow{a_p|\{\eta_k\}\rangle}\right),\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{\eta_p^2}{2E_p}.
\end{aligned}$$

e) To compute the mean square dispersion, let us recall the theorem of elementary probability theory that

$$\langle(\Delta N)^2\rangle = \overline{N^2} - \overline{N}^2.$$

We have already calculated $\overline{N}$ so it is trivial to note that

$$\overline{N}^2 = \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k^2\eta_p^2}{4E_kE_p}.$$

Let us then calculate $\overline{N^2}$.

$$\begin{aligned}
\overline{N^2} = \langle\{\eta_k\}|N^2|\{\eta_k\}\rangle &= \langle\{\eta_k\}|\int \frac{d^3k d^3p}{(2\pi)^6}a_k^\dagger a_k a_p^\dagger a_p|\{\eta_k\}\rangle,\\
&= \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k\eta_p}{2\sqrt{E_kE_p}}\langle\{\eta_k\}|a_k a_p^\dagger|\{\eta_k\}\rangle,\\
&= \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k\eta_p}{2\sqrt{E_kE_p}}\left((2\pi)^3\delta^{(3)}(\vec{k}-\vec{p}) + \langle\{\eta_k\}|a_p^\dagger a_k|\{\eta_k\}\rangle\right),\\
&= \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k} + \int \frac{d^3k d^3p}{(2\pi)^6}\frac{\eta_k^2\eta_p^2}{4E_kE_p}.
\end{aligned}$$

It is therefore quite easy to see that

$$\langle(\Delta N)^2\rangle = \overline{N^2} - \overline{N}^2 = \int \frac{d^3k}{(2\pi)^3}\frac{\eta_k^2}{2E_k}.$$

2. We are given the Lorentz commutation relations,

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho}).$$

a) Given the generators of rotations and boosts defined by,

$$L^i = \frac{1}{2}\epsilon^{ijk}J^{jk} \qquad K^i = J^{0i},$$

we are to explicitly calculate all the commutation relations. We are given trivially that

$$[L^i, L^j] = i\epsilon^{ijk}L^k.$$

Let us begin with the K's. By direct calculation,

$$\begin{aligned}[K^i, K^j] = [J^{0i}, J^{0j}] &= i(g^{0i}J^{0j} - g^{00}J^{ij} - g^{ij}J^{00} + g^{0j}J^{i0}),\\ &= -iJ^{ij};\\ &= -2i\epsilon^{ijk}L^k.\end{aligned}$$

Likewise, we can directly compute the commutator between the L and K's. This also will follow by direct calculation.

$$\begin{aligned}[L^i, K^j] &= \frac{1}{2}\epsilon^{lk}[J^{ilk}, J^{0j}],\\ &= \frac{1}{2}\epsilon^{ilk}i(g^{l0}J^{ij} - g^{i0}J^{lj} - g^{lj}J^{i0} + g^{ij}J^{l0}),\\ &= i\epsilon^{ijk}J^{0k};\\ &= i\epsilon^{ijk}K^k.\end{aligned}$$

We were also to show that the operators

$$J_+^i = \frac{1}{2}(L^i + iK^i) \qquad J_-^i = \frac{1}{2}(L^i - iK^i),$$

could be seen to satisfy the commutation relations of angular momentum. First let us compute,

$$\begin{aligned}[J_+, J_-] &= \frac{1}{4}\left[(L^i + iK^i), (L^j - iK^i)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] - i[L^i, K^j] + [K^i, K^j]\right),\\ &= 0.\end{aligned}$$

In the last line it was clear that I used the commutator $[L^i, K^j]$ derived above. The next two calculations are very similar and there is a lot of 'justification' algebra in each step. There is essentially no way for me to include all of the details of every step, but each can be verified (e.g. $i[K^i, L^j] = -i[L^j, K^i] = (-i)i\epsilon^{jik}K^k = -\epsilon^{ijk}K^k$...*etc*). They are as follows:

$$\begin{aligned}[J_+^i, J_+^j] &= \frac{1}{4}\left[(L^i + iK^i), (L^j + iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] + i[K^i, L^j] + i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k - \epsilon^{ijk}K^k - \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k + iK^k) = i\epsilon^{ijk}J_+^k.\end{aligned}$$

Likewise,

$$\begin{aligned}[J_-^i, J_-^j] &= \frac{1}{4}\left[(L^i - iK^i), (L^j - iK^j)\right],\\ &= \frac{1}{4}\left([L^i, L^j] - i[K^i, L^j] - i[L^i, K^j] + i[L^i, K^i] - [K^i, K^j]\right),\\ &= \frac{1}{4}\left(i\epsilon^{ijk}L^k + \epsilon^{ijk}K^k + \epsilon^{ijk}K^k + i\epsilon^{ijk}L^k\right),\\ &= i\epsilon^{ijk}\frac{1}{2}(L^k - iK^k) = i\epsilon^{ijk}J_-^k.\end{aligned}$$

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b) Let us consider first the $(0, \frac{1}{2})$ representation. For this representation we will need to satisfy

$$J_+^i = \frac{1}{2}(L^i + iK^i) = 0 \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = \frac{\sigma^i}{2}.$$

This is obtained by taking $L^i = \frac{\sigma^i}{2}$ and $K^i = \frac{i\sigma^i}{2}$. The transformation law then of the $(0, \frac{1}{2})$ representation is

$$\begin{aligned}\Phi_{(0,\frac{1}{2})} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(0,\frac{1}{2})},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(0,\frac{1}{2})},\\ &= e^{-\frac{i\theta^i\sigma^i}{2} + \frac{\beta^j K^j}{2}}\Phi_{(0,\frac{1}{2})}.\end{aligned}$$

The calculation for the $(\frac{1}{2}, 0)$ representation is very similar. Taking $L^i = \frac{\sigma^i}{2}$ and $K^i = -\frac{\sigma^i}{2}$, we get

$$J_+^i = \frac{1}{2}(L^i + iK^i) = \frac{\sigma^i}{2} \qquad J_-^i = \frac{1}{2}(L^i - iK^k) = 0.$$

Then the transformation law of the representation is

$$\begin{aligned}\Phi_{(\frac{1}{2},0)} &\longrightarrow e^{-i\omega_{\mu\nu}J^{\mu\nu}}\Phi_{(\frac{1}{2},0)},\\ &= e^{-i(\theta^i L^i + \beta^j K^j)}\Phi_{(\frac{1}{2},0)},\\ &= e^{-\frac{i\theta^i\sigma^i}{2} - \frac{\beta^j K^j}{2}}\Phi_{(\frac{1}{2},0)}.\end{aligned}$$

Comparing these transformation laws with Peskin and Schroeder's (3.37), we see that

$$\psi_L = \Phi_{(\frac{1}{2},0)} \qquad \psi_R = \Phi_{(0,\frac{1}{2})}.$$

3. a) We are given that T_a is a representation of some Lie group. This means that

$$[T_a, T_b] = if^{abc}T_c$$

by definition. Allow me to take the complex conjugate of both sides. Note that $[T_a, T_b] = [(-T_a), (-T_b)]$ in general and recall that f^{abc} are real.

$$\begin{aligned}[T_a, T_b]^* &= (if^{abc}T_c)^*,\\ [T_a^*, T_b^*] &= -if^{abc}T_c^*,\\ \therefore [(-T_a^*), (-T_b^*)] &= if^{abc}(-T_c^*).\end{aligned}$$

So by the definition of a representation, it is clear that $(-T_a^*)$ is also a representation of the algebra.

b) As before, we are given that T_a is a representation of some Lie group. We will take the Hermitian adjoint of both sides.

$$\begin{aligned}[T_a, T_b]^\dagger &= (if^{abc}T_c)^\dagger,\\ (T_aT_b)^\dagger - (T_bT_a)^\dagger &= -if^{abc}T_c^\dagger,\\ T_b^\dagger T_a^\dagger - T_a^\dagger T_b^\dagger &= -if^{abc}T_c^\dagger,\\ [T_b^\dagger, T_a^\dagger] &= -if^{abc}T_c^\dagger,\\ \therefore [T_a^\dagger, T_b^\dagger] &= if^{abc}T_c^\dagger.\end{aligned}$$

So by the definition of a representation, it is clear that $T_a^\dagger$ is a representation of the algebra.

c) We define the spinor representation of $SU(2)$ by $T_a = \frac{\sigma^a}{2}$ so that

$$T_1 \equiv \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad T_2 \equiv \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad T_3 \equiv \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will consider the matrix $S = i\sigma^2$. Clearly S is unitary because $(i\sigma^2)(i\sigma^2)^\dagger = 1$. Now, one could proceed by direct calculation to demonstrate that

$$ST_1S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -T_1^* \qquad ST_2S^\dagger = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -T_2^* \qquad ST_3S^\dagger = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -T_3^*.$$

This clearly demonstrates that the representation $-T_a^*$ is equivalent to that of T_a.

d) From our definitions of our representation of $SO(3,1)$ using J_+^i and J_-^i, it is clear that

$$(J_+^i)^\dagger = J_-^i.$$

This could be expressed as if $(\frac{1}{2},0)^\dagger = (0,\frac{1}{2})$, or, rather $L^\dagger = R$. So what we must ask ourselves is, does there exist a unitary matrix S such that

$$SLS^\dagger = L \qquad \text{but} \qquad SKS^\dagger = -K \text{ ?}$$

If there did exist such a unitary transformation, then we could conclude that L and R are equivalent representations. However, this is not possible in our $SO(3,1)$ representation because both L and K are represented strictly by the Pauli spin matrices so that $iK = L = \frac{\sigma}{2}$. It is therefore clear that there cannot exist a transformation that will change the sign of K yet leave L alone. So the representations are inequivalent.

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